

ANALYSIS OF THE MINIMAL REPRESENTATION OF  $\mathrm{Sp}(r, \mathbb{R})$ 

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**Abstract** The minimal representations of  $\mathrm{Sp}(r, \mathbb{R})$  can be realized on a Hilbert space of holomorphic functions. This is the analogue of the Brylinski-Kostant model. It can also be realized on a Hilbert space of  $L^2$  functions on  $\mathbb{R}^n$ . This is the Schrödinger model. We will describe the two realizations and a transformation which maps one model to the other. It involves the classical Bargmann transform.

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**Introduction.** In the paper [A11] a general construction for a simple complex Lie algebra  $\mathfrak{g}$  and of a real form  $\mathfrak{g}_{\mathbb{R}}$  has been proposed, starting from a pair  $(V, Q)$  where  $V$  is a simple Jordan algebra of rank  $r$  and  $Q$  a polynomial on  $V$ , homogeneous of degree  $2r$ . The Lie algebra  $\mathfrak{g}$  is of Hermitian type. In the paper [A12b], the manifolds  $\Xi$  and  $\Xi^{\sigma}$  are the orbits of the linear form  $\tau_{c_{\lambda}}$  and its conjugate  $\tau_{c_{\lambda}}^{\sigma} = \kappa(\sigma)\tau_{c_{\lambda}}$ , for an idempotent  $c_{\lambda}$ , under the structure group  $\mathrm{Str}(V)$  acting by the restriction of an irreducible representation  $\kappa$  of the conformal group  $\mathrm{Conf}(V, Q)$ . The spaces  $\mathcal{F}(\Xi)$  and  $\mathcal{F}(\Xi^{\sigma})$  are Hilbert spaces of holomorphic functions on the complex manifolds  $\Xi$  and  $\Xi^{\sigma}$ . The minimal representations of  $\mathfrak{g}_{\mathbb{R}}$  are realized in  $\mathcal{F}(\Xi)$  and  $\mathcal{F}(\Xi^{\sigma})$ . In this paper we consider the special case where  $V = \mathrm{Sym}(r, \mathbb{C})$ ,  $Q$  is the square of the determinant and the construction leads to the Lie algebra  $\mathfrak{g} = \mathfrak{sp}(r, \mathbb{C})$  and to a real form  $\mathfrak{g}_{\mathbb{R}} = \mathrm{Ad}(g_0)(\mathfrak{sp}(r, \mathbb{R}))$  for some  $g_0 \in \mathrm{Sp}(r, \mathbb{C})$ . The two minimal representations  $\rho$  and  $\rho^{\sigma}$  of  $\mathfrak{g}_{\mathbb{R}}$  are respectively realized in  $\mathcal{F}(\Xi)$  and  $\mathcal{F}(\Xi^{\sigma})$ , which turn out to be the classical Fock space  $\mathcal{F}(\mathbb{C}^r)$ . They integrate to representations  $T$  and  $T^{\sigma}$  of the complex group  $\mathrm{Mp}(r, \mathbb{C})$  and their restriction to  $\mathrm{Mp}(r, \mathbb{R})$  realize unitary representations in  $T(g_0^{-1})(\mathcal{F}(\mathbb{C}^r))$  and  $T^{\sigma}(g_0^{-1})(\mathcal{F}(\mathbb{C}^r))$ . On another hand, we consider the two minimal representations  $R$  and  $R^{\sigma}$ , i.e. the Segal-Shale-Weil representations, of  $\mathrm{Mp}(r, \mathbb{R})$  on  $L^2(\mathbb{R}^r)$ . We also give unitary integral operators  $\mathcal{B}$  from the space  $L^2(\mathbb{R}^r)$  onto the space  $T(g_0^{-1})(\mathcal{F}(\Xi))$  which intertwins  $\rho = dT$  and  $dR$  of  $\mathfrak{sp}(r, \mathbb{R})$ , and  $\mathcal{B}^{\sigma}$  from the space  $L^2(\mathbb{R}^r)$  onto the space  $T^{\sigma}(g_0^{-1})\mathcal{F}(\Xi^{\sigma})$  which intertwins  $\rho^{\sigma} = dT^{\sigma}$  and  $dR^{\sigma}$  of  $\mathfrak{sp}(r, \mathbb{R})$ .

## 1. The analogue of the Brylinski-Kostant model. General case

Let  $V$  be a simple complex Jordan algebra with rank  $r$  and dimension  $n$  and  $Q$  the homogeneous polynomial of degree  $2r$  on  $V$  given by  $Q(v) = \Delta(v)^2$  where  $\Delta$  is the Jordan algebra determinant. The structure group of  $V$  is defined by:

$$\text{Str}(V) = \{g \in \text{GL}(V) \mid \exists \chi(g) \in \mathbb{C}, \Delta(gz) = \chi(g)\Delta(z)\}$$

The conformal group  $\text{Conf}(V)$  is the group of rational transformations  $g$  of  $V$  generated by: the translations  $z \mapsto z + a$  ( $a \in V$ ), the dilations  $z \mapsto \ell z$  ( $\ell \in \text{Str}(V)$ ), and the conformal inversion  $\sigma : z \mapsto -z^{-1}$  (see [M78]).

Let  $\mathfrak{p}$  be the space of polynomials on  $V$  generated by the polynomials  $Q(z - a)$  of  $Q$ , with  $a \in V$ . Let  $\kappa$  be the cocycle representation of  $K = \text{Conf}(V)$ , defined in [A11] and [AF12] as follows:

$$\begin{aligned} (\kappa(g)p)(z) &= \mu(g^{-1}, z)p(g^{-1}z), \\ \mu(g, z) &= \chi((Dg(z))^{-1}) \quad (g \in K, z \in V). \end{aligned}$$

The function  $\kappa(g)p$  belongs actually to  $\mathfrak{p}$  (see [FG96], Proposition 6.2). The cocycle  $\mu(g, z)$  is a polynomial in  $z$  of degree  $\leq \deg Q$  and

$$\begin{aligned} (\kappa(\tau_a)p)(z) &= p(z - a) \quad (a \in V), \\ (\kappa(\ell)p)(z) &= \chi(\ell)p(\ell^{-1}z) \quad (\ell \in L), \\ (\kappa(\sigma)p)(z) &= Q(z)p(-z^{-1}). \end{aligned}$$

Let  $L = \text{Str}(V)$ . It is established in [A11] that the element  $H_0 \in \mathfrak{z}(\mathfrak{l})$ , given by  $\exp(tH_0) = l_{e^{-t}} : z \in V \mapsto e^{-t}z$ , defines a grading of  $\mathfrak{p}$ :

$$\begin{aligned} \mathfrak{p} &= \mathfrak{p}_{-r} \oplus \mathfrak{p}_{-r+1} \oplus \dots \oplus \mathfrak{p}_0 \oplus \dots \oplus \mathfrak{p}_{r-1} \oplus \mathfrak{p}_r, \\ \mathfrak{p}_j &= \{p \in \mathfrak{p} \mid d\kappa(H_0)p = jp\} \end{aligned}$$

is the set of polynomials in  $\mathfrak{p}$ , homogeneous of degree  $j + r$ . Furthermore  $\kappa(\sigma) : \mathfrak{p}_j \rightarrow \mathfrak{p}_{-j}$ , and  $\mathfrak{p}_{-r} = \mathbb{C}$ ,  $\mathfrak{p}_r = \mathbb{C}Q$ ,  $\mathfrak{p}_{r-1} \simeq V$ ,  $\mathfrak{p}_{-r+1} \simeq V$ . Observe that  $\mathfrak{p}_{r-1} = \{\kappa(\sigma)p \mid p \in \mathfrak{p}_{-r+1}\}$  and that  $\mathfrak{p}_{-r+1}$  is the space of linear forms on  $V$ .

Assume  $r \neq 1$  and denote by  $\mathcal{W} = \mathcal{V} \oplus \mathcal{V}^\sigma$  with  $\mathcal{V} = \mathfrak{p}_{-r+1}$ ,  $\mathcal{V}^\sigma = \mathfrak{p}_{r-1}$ . Let's consider the linear form  $\tau : z \mapsto \text{tr}(z) = \text{trace}(z)$  and its image  $\tau^\sigma$  given by  $\tau^\sigma(z) = Q(z)\tau(-z^{-1})$ . Denote by  $E = \tau$  and  $F = \tau^\sigma$  and let  $X_E \in \mathfrak{k}_1$  and  $X_F \in \mathfrak{k}_{-1}$  such that  $E = d\kappa(X_E)1$  and  $F = d\kappa(X_F)Q$ . Put  $[1, Q] = H_0$ ,  $[E, Q] = -X_E$ ,  $[F, 1] = X_F$ . Let  $\lambda_0$  be a bilinear form on  $\mathcal{V} \times \mathcal{V}^\sigma$ . Then  $\mathfrak{g} = \mathfrak{l} \oplus \mathcal{W}$  carries a unique simple Lie algebra structure such that

- (i)  $[X, X'] = [X, X']_{\mathfrak{l}} \quad (X, X' \in \mathfrak{l})$ ,
- (ii)  $[X, p] = d\kappa(X)p \quad (X \in \mathfrak{l}, p \in \mathcal{W})$ ,
- (iii)  $[E, F] = \lambda_0(E, F)H_0 + [X_E, X_F]$ .

We recall also the real form  $\mathfrak{g}_{\mathbb{R}}$  of  $\mathfrak{g}$  which will be considered in the sequel. We fix a Euclidean real form  $V_{\mathbb{R}}$  of the complex Jordan algebra  $V$ , denote by  $z \mapsto \bar{z}$  the conjugation of  $V$  with respect to  $V_{\mathbb{R}}$ , and then consider the involution  $g \mapsto \bar{g}$  of  $K$  given by:  $\bar{g}z = \overline{g\bar{z}}$ . The involution  $\alpha$  defined by  $\alpha(g) = \sigma\bar{g}\sigma^{-1}$  is a Cartan involution of  $K$  and  $K_{\mathbb{R}} = \{g \in K \mid \alpha(g) = g\}$  is a compact real form of  $K$  and it follows that  $L_{\mathbb{R}} = L \cap K_{\mathbb{R}}$  is a compact real form of  $L$ . Observe that, since for  $g \in \text{Str}(V)$ ,  $\sigma \circ g \circ \sigma = g'$ , the adjoint of  $g$  with respect to the symmetric form  $(w \mid w') = \tau(w\bar{w}')$ , then  $L_{\mathbb{R}} = \{l \in L \mid ll' = \text{id}_V\}$ . Let  $\mathfrak{u}$  be the compact real form of  $\mathfrak{g}$  such that  $\mathfrak{l} \cap \mathfrak{u} = \mathfrak{l}_{\mathbb{R}}$ , the Lie algebra of  $L_{\mathbb{R}}$ . Denote by  $\mathcal{W}_{\mathbb{R}} = \mathfrak{u} \cap (i\mathfrak{u})$ . Then, the real Lie algebra defined by  $\mathfrak{g}_{\mathbb{R}} = \mathfrak{l}_{\mathbb{R}} + \mathcal{W}_{\mathbb{R}}$  is a real form of  $\mathfrak{g}$  and this decomposition is its Cartan decomposition. Since the complexification of the Cartan decomposition of  $\mathfrak{g}_{\mathbb{R}}$  is  $\mathfrak{g} = \mathfrak{l} + \mathcal{W}$  and since  $\mathcal{W} = \mathcal{V} + \mathcal{V}^{\sigma} = \text{d}\kappa(\mathcal{U}(\mathfrak{l}))E + \text{d}\kappa(\mathcal{U}(\mathfrak{l}))F$  is a sum of two simple  $\mathfrak{l}$ -modules, it follows that the simple real Lie algebra  $\mathfrak{g}_{\mathbb{R}}$  is of Hermitian type. One can show that  $\mathcal{W}_{\mathbb{R}} = \{p \in \mathcal{W} \mid \beta(p) = p\}$  where we defined for a polynomial  $p \in \mathcal{W}$ ,  $\bar{p} = \overline{p(\bar{z})}$ , and considered the antilinear involution  $\beta$  of  $\mathcal{W}$  given by  $\beta(p) = \kappa(\sigma)\bar{p}$ .

Consider the isomorphisms  $v \in V \mapsto \tau_v \in \mathcal{V}$  and  $v \in V \mapsto \tau_v^{\sigma} \in \mathcal{V}^{\sigma}$  where  $\tau_v$  is the linear form on  $V$  given by  $\tau_v(v') = \tau(vv')$  and  $\tau_v^{\sigma} = \kappa(\sigma)\tau_v$ . For a suitable choice of a bilinear form  $\lambda_0 : \mathcal{V} \times \mathcal{V}^{\sigma} \rightarrow \mathbb{C}$ ,  $\mathfrak{g}$  and  $\mathfrak{g}_{\mathbb{R}}$  are isomorphic to matrix Lie algebras ([A12b]). Let  $\mathfrak{h}$  be a Cartan subalgebra of  $\mathfrak{l}$ , which contains  $H_0$ . Since the Lie algebra  $\mathfrak{g} = \mathfrak{l} + \mathcal{W}$  is of Hermitian type, then  $\mathfrak{h}$  is also a Cartan subalgebra of  $\mathfrak{g}$ . Let  $\Delta^+(\mathfrak{g}, \mathfrak{h})$  be a suitably chosen positive system of roots and let  $\lambda$  be the highest root. Denote by  $\mathfrak{g}_{\lambda}$  and  $\mathfrak{g}_{-\lambda}$  the root spaces corresponding to the highest and lowest root. One knows that the minimal adjoint nilpotent orbit of  $\mathfrak{g}$  is given by  $\mathcal{O}_{\min} = G \cdot (\mathfrak{g}_{\lambda} \setminus \{0\})$ , where  $G = \text{Int}(\mathfrak{g})$  is the group of inner automorphisms of  $\mathfrak{g}$ . Recall from [A12b], Lemma 2.1, that  $\mathfrak{g}_{\lambda} \subset \mathcal{V} \setminus \{0\}$  and  $\mathfrak{g}_{-\lambda} = \{\kappa(\sigma)T \mid T \in \mathfrak{g}_{\lambda}\} \subset \mathcal{V}^{\sigma} \setminus \{0\}$ . Denote by  $\tau_{\lambda}$  the linear form given by  $\tau_{\lambda}(v) = \tau_{c_{\lambda}}(v) = \tau(c_{\lambda}v)$ , and by  $\tau_{\lambda}^{\sigma} = \kappa(\sigma)\tau_{\lambda}$ . They are nilpotent elements of  $\mathfrak{g}$ ,  $\tau_{\lambda} \in \mathfrak{g}_{\lambda}$  and  $\tau_{\lambda}^{\sigma} \in \mathfrak{g}_{-\lambda}$ . The orbits  $\Xi$  of  $\tau_{\lambda}$  and  $\Xi^{\sigma}$  of  $\tau_{\lambda}^{\sigma}$  under the group  $L$  acting on  $\mathfrak{p}$  by the representation  $\kappa$ , are conical varieties related by  $\Xi^{\sigma} = \kappa(\sigma)\Xi$ . They are the minimal nilpotent  $L$ -orbits in  $\mathcal{W}$ . Polynomials  $\xi \in \Xi$  and  $\xi^{\sigma} = \kappa(\sigma)\xi \in \Xi^{\sigma}$  can be written

$$\begin{aligned}\xi(v) &= \kappa(l)\tau_{\lambda}(v) = \chi(l)\tau((l')^{-1}c_{\lambda}v), \\ \xi^{\sigma}(v) &= \kappa(\sigma)\xi(v) = \chi(l)Q(v)\tau(-((l')^{-1}c_{\lambda})v^{-1}),\end{aligned}$$

where  $l' \in \text{Str}(V)$  is the adjoint of  $l$  for the inner product  $(x \mid y) = \text{tr}(xy)$ . Then  $\Xi$  and  $\Xi^{\sigma}$  are realized as a  $L$ -orbit of  $c_{\lambda}$  in  $V$  and have explicit coordinate systems.

We consider in this paper the case of  $V_{\mathbb{R}} = \text{Sym}(r, \mathbb{R})$ ,  $V = \text{Sym}(r, \mathbb{C})$ ,  $\text{Str}(V_{\mathbb{R}}) = \text{GL}(r, \mathbb{R})$ ,  $\text{Str}(V) = \text{GL}(r, \mathbb{C})$  acting on  $V_{\mathbb{R}}$  or on  $V$  by  $gx = g \cdot x \cdot g^t$ . Then  $c_{\lambda} = \text{diag}(1, 0, \dots, 0)$ , the orbits  $\Xi$  and  $\Xi^{\sigma}$  are both identified to

$$\Gamma(\Xi) = \{g \cdot c_{\lambda} \cdot g^t \mid g \in \text{GL}(r, \mathbb{C})\} = \{\xi_z = zz^t \mid z \in \mathbb{C}^r\}.$$

The group  $L$  acts on the spaces  $\mathcal{O}(\Xi)$  and  $\mathcal{O}(\Xi^{\sigma})$  of holomorphic functions on  $\Xi$  and  $\Xi^{\sigma}$  respectively by:

$$(\pi_{\alpha}(l)f)(\xi) = \chi(l)^{\alpha} f(\kappa(l)^{-1}\xi) \text{ and } (\pi_{\alpha}^{\sigma}(l)f)(\xi^{\sigma}) = \chi(l)^{\alpha} f(\kappa(l)^{-1}\xi^{\sigma}).$$

For  $\xi = X_z \in \Gamma(\Xi)$  and for every function  $f \in \mathcal{O}(\Xi)$ , we write  $f(\xi) = \phi(z)$ . In these coordinates, the representations  $\pi_{\alpha}$  and  $\pi_{\alpha}^{\sigma}$  are given by

$$\pi_{\alpha}(l)\phi(z) = \chi(l)^{\alpha} \phi(l_1^{-1} \cdot z) \text{ for } l = (l_1, (l_1^t)^{-1})$$

and

$$\pi_{\alpha}^{\sigma}(l)\phi(z) = \chi(l)^{-\alpha} \phi(l_1^t \cdot z) \text{ for } l = (l_1, (l_1^t)^{-1}).$$

For  $m \in \mathbb{Z}$ , let  $\mathcal{O}_m(\Xi)$  and  $\mathcal{O}_m(\Xi^{\sigma})$  be the the spaces of holomorphic functions  $f$  on  $\Xi$  and  $f^{\sigma}$  on  $\Xi^{\sigma}$  respectively such that for every  $w \in \mathbb{C}^*$ ,

$$f(w\xi) = w^m f(\xi) \text{ and } f^{\sigma}(w\xi^{\sigma}) = w^m f^{\sigma}(\xi^{\sigma}).$$

These spaces are respectively invariant under  $\pi_{\alpha}$  and  $\pi_{\alpha}^{\sigma}$ . For  $f \in \mathcal{O}_m(\Xi)$ ,  $h \in \mathcal{O}_{m+\frac{1}{2}}(\Xi)$  and  $f^{\sigma} \in \mathcal{O}_m(\Xi^{\sigma})$ ,  $h^{\sigma} \in \mathcal{O}_{m+\frac{1}{2}}(\Xi^{\sigma})$  their corresponding functions  $\phi$ ,  $\psi$  and  $\phi^{\sigma}$ ,  $\psi^{\sigma}$  on  $\mathbb{C}^r$  satisfy for  $\mu > 0$ ,

$$\begin{aligned} \phi(\mu \cdot (z)) &= \mu^{2m} \phi(z), & \psi(\mu \cdot (z)) &= \mu^{2m+1} \psi(z), \\ \phi^{\sigma}(\mu \cdot (z)) &= \mu^{2m} \phi^{\sigma}(z), & \psi^{\sigma}(\mu \cdot (z)) &= \mu^{2m+1} \psi^{\sigma}(z), \end{aligned}$$

and the correspondances  $f(\xi) \mapsto \phi(z)$  and  $f^{\sigma}(\xi^{\sigma}) \mapsto \phi^{\sigma}(z)$  map the spaces  $\mathcal{O}_m(\Xi)$ ,  $\mathcal{O}_m(\Xi^{\sigma})$  and  $\mathcal{O}_{m+\frac{1}{2}}(\Xi)$ ,  $\mathcal{O}_{m+\frac{1}{2}}(\Xi^{\sigma})$ , to respectively

$$\mathcal{O}_{2m}(\mathbb{C}^r) = \{\phi \in \mathcal{O}(\mathbb{C}^r) \mid \phi(\mu z) = \mu^{2m} \phi(z)\},$$

$$\mathcal{O}_{2m+1}(\mathbb{C}^r) = \{\phi \in \mathcal{O}(\mathbb{C}^r) \mid \phi(\mu z) = \mu^{2m+1} \phi(z)\}.$$

Let  $\tilde{\mathcal{O}}_{2m}(\mathbb{C}^{d_0})$ ,  $\tilde{\mathcal{O}}_{2m+1}(\mathbb{C}^{d_0})$  and  $\tilde{\mathcal{O}}_{2m}^{\sigma}(\mathbb{C}^{d_0})$ ,  $\tilde{\mathcal{O}}_{2m+1}^{\sigma}(\mathbb{C}^{d_0})$  be the sets of such functions  $\phi$ ,  $\phi^{\sigma}$  and  $\psi$ ,  $\psi^{\sigma}$  corresponding to the functions  $f \in \mathcal{O}_m(\Xi)$ ,  $f^{\sigma} \in \mathcal{O}_m(\Xi^{\sigma})$ ,  $h \in \mathcal{O}_{m+\frac{1}{2}}(\Xi)$ ,  $h^{\sigma} \in \mathcal{O}_{m+\frac{1}{2}}(\Xi^{\sigma})$ .

Denote by  $\pi_{\alpha, m}$ ,  $\pi_{\alpha, m+\frac{1}{2}}$  and  $\pi_{\alpha, m}^{\sigma}$ ,  $\pi_{\alpha, m+\frac{1}{2}}^{\sigma}$  the restrictions of the representations  $\pi_{\alpha}$  and  $\pi_{\alpha}^{\sigma}$  to the spaces  $\mathcal{O}_m(\Xi)$ ,  $\mathcal{O}_{m+\frac{1}{2}}(\Xi)$  and  $\mathcal{O}_m(\Xi^{\sigma})$ ,  $\mathcal{O}_{m+\frac{1}{2}}(\Xi^{\sigma})$  and also by  $\pi_{\alpha, 2m}$ ,  $\pi_{\alpha, 2m}^{\sigma}$ ,  $\pi_{\alpha, 2m+1}$ ,  $\pi_{\alpha, 2m+1}^{\sigma}$  the corresponding representations on  $\tilde{\mathcal{O}}_{2m}(\mathbb{C}^r)$ ,  $\tilde{\mathcal{O}}_{2m+1}(\mathbb{C}^r)$ ,  $\tilde{\mathcal{O}}_{2m}^{\sigma}(\mathbb{C}^r)$ ,  $\tilde{\mathcal{O}}_{2m+1}^{\sigma}(\mathbb{C}^r)$ . It follows from Theorem 3.1. in [A.12b] that these spaces consist in polynomials and are finite dimensional,  $\mathcal{O}_m(\Xi) = \{0\}$  and  $\mathcal{O}_m(\Xi^{\sigma}) = \{0\}$  for  $m < 0$ , and  $\pi_{\alpha, 2m}$ ,  $\pi_{\alpha, 2m+1}$ ,  $\pi_{\alpha, 2m}^{\sigma}$ ,  $\pi_{\alpha, 2m+1}^{\sigma}$  are irreducible.

Also,  $L_{\mathbb{R}}$ -invariant norms on  $\mathcal{O}_m(\Xi)$ ,  $\mathcal{O}_{m+\frac{1}{2}}(\Xi)$  and  $\mathcal{O}_m(\Xi^{\sigma})$ ,  $\mathcal{O}_{m+\frac{1}{2}}(\Xi^{\sigma})$  are defined by:

$$\|\phi\|_m^2 = \frac{1}{a_m} \int_{\mathbb{C}^r} |\phi(z)|^2 H(z)^{-2m} m_0(dz),$$

$$\|\phi\|_{m+\frac{1}{2}}^2 = \frac{1}{a_{m+\frac{1}{2}}} \int_{\mathbb{C}^r} |\phi(z)|^2 H(z)^{-2m+1} m_0(dz),$$

where

$$H(z) = \tau(\frac{1}{r}I_r + z\bar{z}) = 1 + \text{tr}(z\bar{z}),$$

and  $m_0(d(z)) = H(z)^{-(r+1)}m(d(z))$ , is the  $L$ -invariant measure,  $p_m$  and  $p_{m+\frac{1}{2}}$  are suitable integers, and

$$a_m = \int_{\mathbb{C}^r} H(z)^{-2m} m_0(dz), \quad a_{m+\frac{1}{2}} = \int_{\mathbb{C}^r} H(z)^{-(2m+1)} m_0(dz).$$

$$a_m = \pi^r \frac{1}{(2m+d_0)\dots(2m+1)}, \quad a_{m+\frac{1}{2}} = \pi^r \frac{1}{(2m+1+d_0)\dots(2m+2)}.$$

These spaces become invariant Hilbert spaces with reproducing kernels:

$$\mathcal{K}_m(\xi, \xi') = \Phi(\xi, \xi')^{2m}, \quad \mathcal{K}_{m+\frac{1}{2}}(\xi, \xi') = \Phi(\xi, \xi')^{2m+1},$$

with

$$\Phi(\xi, \xi') = \tau(\frac{1}{r}I_r + z\bar{z}') \text{ for } \xi = \xi_z, \xi' = \xi_{z'}.$$

They are the irreducible  $L_{\mathbb{R}}$ -invariant subspaces of  $\mathcal{O}(\Xi)$  and  $\mathcal{O}(\Xi^\sigma)$ . The Fock spaces  $\mathcal{F}(\Xi)$  and  $\mathcal{F}(\Xi^\sigma)$  are Hilbert subspaces of  $\mathcal{O}(\Xi)$  and  $\mathcal{O}(\Xi)$  which are  $L$ -invariant, they therefore decompose

$$\mathcal{F}(\Xi) = \sum_{m=0}^{\infty} \mathcal{O}_m(\Xi) + \sum_{m=0}^{\infty} \mathcal{O}_{m+\frac{1}{2}}(\Xi),$$

$$\mathcal{F}(\Xi^\sigma) = \sum_{m=0}^{\infty} \mathcal{O}_m(\Xi^\sigma) + \sum_{m=0}^{\infty} \mathcal{O}_{m+\frac{1}{2}}(\Xi^\sigma).$$

The Hilbert norms on  $\mathcal{F}(\Xi)$  and  $\mathcal{F}(\Xi)$  are of the following form : for

$$f = \sum_{m=0}^{\infty} f_m + \sum_{m=0}^{\infty} f_{m+\frac{1}{2}}$$

then

$$\|f\|_{\mathcal{F}}^2 = \sum_{m=0}^{\infty} \frac{1}{c_m} \|f_m\|_m^2 + \sum_{m=0}^{\infty} \frac{1}{c_{m+\frac{1}{2}}} \|f_{m+\frac{1}{2}}\|_{m+\frac{1}{2}}^2,$$

The sequences  $(c_m)$ ,  $(c_{m+\frac{1}{2}})$ , are determined in such a way that the representation  $\rho_\alpha$  of  $\mathfrak{g}_{\mathbb{R}}$  and  $\rho_\alpha^\sigma$  are unitary (see [A12b], Theorem 5.1). One gets

$$c_m = \frac{1}{(2m)!}, \quad c_{m+\frac{1}{2}} = \frac{1}{(2m+1)!},$$

and then  $\mathcal{F}(\Xi)$  and  $\mathcal{F}(\Xi)$  turn out to be the classical Fock space  $\mathcal{F}(\mathbb{C}^r)$ .

For the representations  $\rho_\alpha$  and  $\rho_\alpha^\sigma$  of  $\mathfrak{g}$ , the elements  $\omega \in \mathfrak{l}$  act by

$$\rho_\alpha(\omega) = d\pi_\alpha(\omega) - \frac{1}{2}d\pi_\alpha(H_0)$$

and

$$\rho_\alpha^\sigma(\omega) = d\pi_\alpha^\sigma(\omega) - \frac{1}{2}d\pi_\alpha^\sigma(H_0).$$

For  $\rho_\alpha$ , the elements  $p \in \mathcal{V}$  act by multiplication and the elements  $p^\sigma \in \mathcal{V}^\sigma$  act by differentiation, and, for  $\rho_\alpha^\sigma = \pi(\sigma)\rho_\alpha\pi(\sigma)$ , the elements  $p \in \mathcal{V}$  act by differentiation and the elements  $p^\sigma \in \mathcal{V}^\sigma$  act by multiplication.

The representation  $\rho_\alpha$  is determined by the operators  $\rho_\alpha(E)$  which involves the multiplication operator  $\tau(z^2)$ , and  $\rho_\alpha(F) = -\rho_\alpha(E)^*$ . The representation  $\rho_\alpha^\sigma$  is determined by the operators  $\rho_\alpha^\sigma(E)$  which involves the differential operator  $\tau(\frac{\partial^2}{\partial z^2})$ , and  $\rho_\alpha(F) = -\rho_\alpha(E)^*$ .

The operators  $\rho_\alpha(E), \rho_\alpha(F), \rho_\alpha(H_0)$  and  $\rho_\alpha^\sigma(E), \rho_\alpha^\sigma(F), \rho_\alpha^\sigma(H_0)$  are given by

$$\rho_\alpha(E)\phi(z) = \frac{i}{4}\tau(z^2)\phi(z),$$

$$\rho_\alpha^\sigma(E)\phi(z) = \frac{i}{4}\tau(\frac{\partial^2}{\partial z^2})\phi(z)$$

$$\rho_\alpha(F)\phi(z) = \frac{i}{4}\tau(\frac{\partial^2}{\partial z^2})\phi(z),$$

$$\rho_\alpha^\sigma(F)\phi(z) = \frac{i}{4}\tau(z^2)\phi(z),$$

$$\rho_\alpha(H_0)\phi(z, z') = (1 - r)(-\alpha r\phi(z, z') + \frac{1}{2}\mathcal{E}\phi(z, z')),$$

$$\rho_\alpha^\sigma(H_0)\phi(z, z') = (1 - r)(\alpha r\phi(z, z') + \frac{1}{2}\mathcal{E}\phi(z, z')),$$

where  $\alpha = -\frac{1}{4}$  and  $\mathcal{E}$  is the Euler operator given by

$$(\mathcal{E}\phi)(z) = \frac{d}{ds}\bigg|_{s=1} \phi(sz).$$

## 2. Some harmonic analysis

We consider on  $V = \text{Sym}(r, \mathbb{C})$  the homogeneous polynomial

$$Q(x) = \det(x)^2.$$

Then the structure group is

$$L = \text{Str}(V, \Delta) = \text{GL}(r, \mathbb{C})$$

(quotiented by  $\{\pm I_r\}$ ). The orbit  $\Xi$  of  $\tau_\lambda$  under  $L$  has dimension  $r$  and can be identified to

$$\Gamma(\Xi) = \{\xi_z = zz^t \mid z \in \mathbb{C}^r\}.$$

Let  $\mathcal{Y}_m(\mathbb{R}^r)$  be the space of spherical harmonics of degree  $m$  on  $\mathbb{R}^r$ : harmonic polynomials which are homogeneous of degree  $m$  on  $\mathbb{R}^r$ . The map

$$\mathcal{Y}_m(\mathbb{R}^r) \rightarrow \mathcal{O}_m(\Xi), \quad \Phi \mapsto f,$$

given by

$$f(\xi_z) = \int_S \langle z, x \rangle^m \Phi(x) s(dx),$$

where

$$\langle z, x \rangle = \sum_{j=1}^r z_j x_j,$$

is an isomorphism which intertwines the representations of  $O(r)$  on both spaces. ( $S$  is the unit sphere in  $\mathbb{R}^r$ , and  $s(dx)$  is the uniform measure on  $S$  with total measure equal to one (see [F.15], section 2).

For a holomorphic function  $a$  on  $\mathbb{C}$  we define the integral operator  $A$  from  $\mathcal{C}(S)$  into  $\mathcal{O}(\Xi)$ :

$$Af(\xi_z) = \int_S a(\langle z, x \rangle) f(x) s(dx).$$

The operator  $A$  is equivariant with respect to the action of  $O(r)$  and maps  $\mathcal{Y}_m(\mathbb{R}^r)$  into  $\mathcal{O}_m(\Xi)$ .

### 3. The Lie algebra $\mathfrak{g}$ and its isomorphism with $\mathfrak{sp}(r, \mathbb{C})$

The Lie algebra  $\mathfrak{g}$  is isomorphic to the matrix Lie algebra

$$\widetilde{\mathfrak{g}} = \left\{ \begin{pmatrix} \omega_1 & v \\ u & -\omega_1^t \end{pmatrix} \mid u, v \in \text{Sym}(r, \mathbb{C}), \quad \omega_1 \in \mathfrak{gl}(r, \mathbb{C}) \right\},$$

and the isomorphism is given by

$$\tau_u + \omega + \tau_v^\sigma \in \mathfrak{g} = \mathcal{V} \oplus \mathfrak{l} \oplus \mathcal{V}^\sigma \mapsto \widetilde{\tau}_u + \widetilde{\omega} + \widetilde{\tau}_v^\sigma \in \widetilde{\mathfrak{g}}$$

with  $\omega = (\omega_1, -\omega_1^t)$ ,

$$\begin{aligned} \widetilde{\omega} &= \begin{pmatrix} \omega_1 & 0 \\ 0 & -\omega_1^t \end{pmatrix} + \text{tr}(\omega_1) \begin{pmatrix} -I_r & 0 \\ 0 & I_r \end{pmatrix}, \\ \widetilde{\tau}_u &= \frac{1}{2} \begin{pmatrix} 0 & 0 \\ u & 0 \end{pmatrix}, \quad \widetilde{\tau}_v^\sigma = \frac{1}{2} \begin{pmatrix} 0 & v^t \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Then,  $\mathfrak{g}$  is isomorphic to  $\mathfrak{sp}(r, \mathbb{C})$ . In fact this follows from Proposition 1.1. in[A12b]: every  $\omega = (\omega_1, -\omega_1^t) \in \mathfrak{l}$  acts on  $V$  by  $\omega x = \omega_1 \cdot x + x \cdot \omega_1^t$ . It follows that for every  $\omega, \omega' \in \mathfrak{l}$ , one has  $[\widetilde{\omega}, \widetilde{\omega'}] = [\widetilde{\omega}, \widetilde{\omega'}]$ . Moreover, for  $\omega = (\omega_1, -\omega_1^t) \in \mathfrak{l}$ , one has

$$[\omega, \tau_u] = d\kappa(\omega)\tau_u = \tau_{-\omega_1^t u - u\omega_1 + 2(\text{tr}(\omega_1))u}$$

and

$$[\widetilde{\omega}, \widetilde{\tau}_u] = \frac{1}{2} \begin{pmatrix} 0 & 0 \\ -\omega_1^t u - u\omega_1 + 2\text{tr}(\omega_1)u & 0 \end{pmatrix}$$

then

$$[\widetilde{\omega}, \widetilde{\tau}_u] = [\widetilde{\omega}, \widetilde{\tau}_u].$$

One has also

$$[\omega, \tau_v^\sigma] = d\kappa(\omega)\tau_v^\sigma = \tau_{\omega_1^t v + v\omega_1^t + 2\text{tr}(\omega_1)v}$$

and

$$[\widetilde{\omega}, \widetilde{\tau}_v^\sigma] = \frac{1}{2} \begin{pmatrix} 0 & v^t \omega_1^t + \omega_1 v^t + 2\text{tr}(\omega_1)v^t \\ 0 & 0 \end{pmatrix}$$

then

$$[\widetilde{\omega}, \widetilde{\tau}_v^\sigma] = [\widetilde{\omega}, \widetilde{\tau}_v^\sigma].$$

The matrices corresponding to  $E = \tau$ ,  $F = \tau^\sigma$  and  $H_0$  (in (i) and (ii)) are

$$\widetilde{E} = \frac{1}{2} \begin{pmatrix} 0 & 0 \\ I_r & 0 \end{pmatrix}, \quad \widetilde{F} = \frac{1}{2} \begin{pmatrix} 0 & I_r \\ 0 & 0 \end{pmatrix}, \quad \widetilde{H}_0 = (1-r) \begin{pmatrix} -\frac{1}{2}I_r & 0 \\ 0 & \frac{1}{2}I_r \end{pmatrix}$$

then  $[\widetilde{E}, \widetilde{F}] = \frac{1}{2(1-r)}\widetilde{H}_0$  and, since  $[E, F] = (\lambda_0(E, F) + \frac{1}{2})H_0 = (\frac{1}{2} - \frac{r}{2(r-1)})H_0$ , then  $[\widetilde{E}, \widetilde{F}] = [\widetilde{E}, \widetilde{F}]$ . Observe that

$$[\widetilde{H}_0, \widetilde{E}] = -(r-1)\widetilde{E} \text{ and } [\widetilde{H}_0, \widetilde{F}] = (r-1)\widetilde{F},$$

i.e.

$$[\widetilde{H}_0, \widetilde{E}] = [\widetilde{H}_0, \widetilde{E}] \text{ and } [\widetilde{H}_0, \widetilde{F}] = [\widetilde{H}_0, \widetilde{F}].$$

This proves that we have obtained an explicit Lie algebra isomorphism from  $\mathfrak{g}$  to  $\widetilde{\mathfrak{g}} = \mathfrak{sp}(r, \mathbb{C})$ .



Now, let's consider the image  $\widetilde{\mathfrak{g}}_{\mathbb{R}}$  of the real form  $\mathfrak{g}_{\mathbb{R}}$  by this isomorphism. Since  $\mathfrak{g}_{\mathbb{R}}$  is given by  $\mathfrak{g}_{\mathbb{R}} = \mathfrak{l}_{\mathbb{R}} + \mathcal{W}_{\mathbb{R}}$ , where  $\mathfrak{l}_{\mathbb{R}}$  is the compact real form of  $\mathfrak{l}$  and  $\mathcal{W}_{\mathbb{R}}$  is generated by the elements  $\tau_u + \tau_u^{\sigma}$  and  $i(\tau_u - \tau_u^{\sigma})$  for  $u \in V_{\mathbb{R}} = \text{Sym}(\mathbb{R})$ , then  $\widetilde{\mathfrak{g}}_{\mathbb{R}}$  is given by  $\widetilde{\mathfrak{g}}_{\mathbb{R}} = \widetilde{\mathfrak{l}}_{\mathbb{R}} + \widetilde{\mathcal{W}}_{\mathbb{R}}$  with

$$\widetilde{\mathfrak{l}}_{\mathbb{R}} = \left\{ \begin{pmatrix} \omega_1 & 0 \\ 0 & -\omega_1^t \end{pmatrix} \mid \omega_1 = w_1 + iw'_1, w_1 \in \text{Skew}(r, \mathbb{R}), w'_1 \in \text{Sym}(r, \mathbb{R}) \right\},$$

$$\widetilde{\mathcal{W}}_{\mathbb{R}} = \left\{ \begin{pmatrix} 0 & u - iv \\ u + iv & 0 \end{pmatrix} \mid u, v \in \text{Sym}(r, \mathbb{R}) \right\},$$

i.e.

$$\widetilde{\mathfrak{g}}_{\mathbb{R}} = \left\{ \begin{pmatrix} \omega_1 & u - iv \\ u + iv & -\omega_1^t \end{pmatrix} \mid \omega_1 = w_1 + iw'_1 \in \mathfrak{su}(r), u, v \in \text{Sym}(r, \mathbb{R}) \right\}.$$

Let  $g_0$  be the element of  $\text{Sp}(r, \mathbb{C})$  given by

$$g_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} iI_r & I_r \\ -I_r & -iI_r \end{pmatrix}.$$

Since

$$\text{Ad}(g_0) \begin{pmatrix} w_1 & w'_1 \\ -w'_1 & w_1 \end{pmatrix} = \begin{pmatrix} w_1 + iw'_1 & 0 \\ 0 & w_1 - iw'_1 \end{pmatrix}$$

and

$$\text{Ad}(g_0) \begin{pmatrix} v & -u \\ -u & -v \end{pmatrix} = \begin{pmatrix} 0 & u - iv \\ u + iv & 0 \end{pmatrix},$$

then

$$\text{Ad}(g_0^{-1})(\widetilde{\mathfrak{l}}_{\mathbb{R}}) = \left\{ \begin{pmatrix} w_1 & w'_1 \\ -w'_1 & w_1 \end{pmatrix} \mid w_1 \in \text{Skew}(r, \mathbb{R}), w'_1 \in \text{Sym}(r, \mathbb{R}) \right\},$$

and

$$\text{Ad}(g_0^{-1})(\widetilde{\mathcal{W}}_{\mathbb{R}}) = \left\{ \begin{pmatrix} v & -u \\ -u & -v \end{pmatrix} \mid u, v \in \text{Sym}(r, \mathbb{R}) \right\},$$

in such a way that

$$\begin{aligned} \text{Ad}(g_0^{-1})(\widetilde{\mathfrak{g}}_{\mathbb{R}}) &= \\ \left\{ \begin{pmatrix} w_1 + v & w'_1 - u \\ -w'_1 - u & w_1 - v \end{pmatrix}, w_1 \in \text{Skew}(r, \mathbb{R}), w'_1, u, v \in \text{Sym}(r, \mathbb{R}) \right\} &= \mathfrak{sp}(r, \mathbb{R}). \end{aligned}$$

Furthermore, it is well known that the minimal nilpotent  $\widetilde{G}$ -orbit in  $\widetilde{\mathfrak{g}}$  is

$$\widetilde{\mathcal{O}}_{\min} = \text{Ad}(\widetilde{G})(e_{11})$$

where  $E_{11}$  is the diagonal matrix  $E_{11} = \text{diag}(1, 0, \dots, 0)$ . and  $e_{11} = \begin{pmatrix} 0 & E_{11} \\ 0 & 0 \end{pmatrix} = 2\tau_{c_\lambda}^\sigma$  is the highest root. It follows that

$$\widetilde{\mathcal{O}}_{\min} \cap \widetilde{\mathcal{W}} = \widetilde{\mathcal{O}}_{\min} \cap \widetilde{\mathcal{V}} \cup \widetilde{\mathcal{O}}_{\min} \cap \widetilde{\mathcal{V}}^\sigma$$

with

$$\begin{aligned} \widetilde{\mathcal{O}}_{\min} \cap \widetilde{\mathcal{V}}^\sigma &= \text{Ad}(\widetilde{L})(e_{11}) \\ &= \{ \text{Ad}(l)(e_{11}) \mid l = \begin{pmatrix} l_1 & 0 \\ 0 & (l_1^t)^{-1} \end{pmatrix} \mid l_1 \in \text{GL}(r, \mathbb{C}) \} \\ &= \left\{ \begin{pmatrix} 0 & l_1 E_{11} l_1^t \\ 0 & 0 \end{pmatrix} \mid l_1 \in \text{GL}(r, \mathbb{C}) \right\}, \\ &= \left\{ \begin{pmatrix} 0 & z z^t \\ 0 & 0 \end{pmatrix} \mid z \in \mathbb{C}^r \right\} \end{aligned}$$

and

$$\begin{aligned} \widetilde{\mathcal{O}}_{\min} \cap \mathcal{V} &= \text{Ad}(\widetilde{L})(\text{Ad}(J)e_{11}) \\ &= \{ \text{Ad}(l)(\text{Ad}(J)e_{11}) \mid l = \begin{pmatrix} l_1 & 0 \\ 0 & (l_1^t)^{-1} \end{pmatrix} \mid l_1 \in \text{GL}(r, \mathbb{C}) \} \\ &= \left\{ \begin{pmatrix} 0 & 0 \\ l_1 E_{11} l_1^t & 0 \end{pmatrix} \mid l_1 \in \text{GL}(r, \mathbb{C}) \right\} \\ &= \left\{ \begin{pmatrix} 0 & 0 \\ z z^t & 0 \end{pmatrix} \mid z \in \mathbb{C}^r \right\}. \end{aligned}$$

It follows that  $\widetilde{\mathcal{O}}_{\min} \cap \widetilde{\mathcal{V}}^\sigma$  and  $\widetilde{\mathcal{O}}_{\min} \cap \mathcal{V}$  are respectively the images of the orbits  $\Xi$  and  $\Xi^\sigma$  by the isomorphism  $\mathfrak{g} \rightarrow \widetilde{\mathfrak{g}}$  and are diffeomorphic to

$$\Gamma(\Xi) = \{ \xi_z = z z^t \mid z \in \mathbb{C}^r \}$$

and that the map  $\kappa(\sigma)$  corresponds here to  $X \mapsto \text{Ad}(J)X$ .

Moreover,

$$\widetilde{\mathcal{O}}_{\min} \cap \mathfrak{sp}(r, \mathbb{R}) = \widetilde{\mathcal{O}}_{\min} \cap \text{Ad}(g_0^{-1})(\widetilde{\mathfrak{g}}_{\mathbb{R}}) = Y^+ \cup Y^-.$$

#### 4. The Schrödinger model.

Let  $\Gamma_{\mathbb{R}}$  be the open cone in  $\mathbb{R}^r$  and  $S$  be the unit sphere given respectively by:

$$\Gamma_{\mathbb{R}} = \{x \in \mathbb{R}^r \mid |x| \neq 0\},$$

and

$$S = \{x \in \mathbb{R}^r : |x| = 1\}.$$

The group  $L_{\mathbb{R}} = \text{GL}(r, \mathbb{R})$  acts on  $\mathbb{R}^r$  by the natural representation. This action stabilizes the cone  $\Gamma_{\mathbb{R}}$ . The multiplicative group  $\mathbb{R}_+^*$  acts on  $\Gamma_{\mathbb{R}}$  as a dilation and the quotient space  $M = \Gamma_{\mathbb{R}}/\mathbb{R}_+^*$  is identified with  $S$ . This defines an action of  $L_{\mathbb{R}}$  on  $S$ , which leads to a  $L_{\mathbb{R}}$ -equivariant principal  $\mathbb{R}_+^*$ -bundle:  $\Gamma_{\mathbb{R}} \rightarrow S, x \mapsto \frac{x}{|x|}$ . For  $\lambda \in \mathbb{C}$ , let  $\mathcal{E}_{\lambda}(\Gamma_{\mathbb{R}})$  and  $\mathcal{E}_{\lambda}^{\sigma}(\Gamma_{\mathbb{R}})$  be the spaces of  $\mathcal{C}^{\infty}$ -functions on  $\Gamma_{\mathbb{R}}$  homogeneous of degree  $\lambda$ :

$$\mathcal{E}_{\lambda}(\Gamma_{\mathbb{R}}) = \mathcal{E}_{\lambda}^{\sigma}(\Gamma_{\mathbb{R}}) = \{f \in \mathcal{C}^{\infty}(\Gamma_{\mathbb{R}}) \mid f(tx) = t^{\lambda}f(x), \quad x \in \Gamma, t > 0\},$$

The group  $L_{\mathbb{R}} = \text{GL}(r, \mathbb{R})$  acts naturally on  $\mathcal{E}_{\lambda}(\Gamma_{\mathbb{R}})$ , and, under the action of the subgroup  $O(r)$ , the space  $\mathcal{E}_{\lambda}(\Gamma_{\mathbb{R}})$  decomposes as:

$$\mathcal{E}_{\lambda}(\Gamma_{\mathbb{R}}) \big|_S \simeq \bigoplus_{k=0}^{\infty} \mathcal{Y}_k(\mathbb{R}^r).$$

These representations extend to representations  $R$  and  $R^{\sigma}$  of the metaplectic group  $\text{Mp}(r, \mathbb{R})$  on the Hilbert space  $L^2(\mathbb{R}^r)$  as follows: denote by

$$g(l_1) = \begin{pmatrix} l_1 & 0 \\ 0 & (l_1^t)^{-1} \end{pmatrix} \text{ for } l_1 \in \text{GL}(r, \mathbb{R}),$$

$$t(u) = \exp(2\widetilde{\tau}_u) = \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}, \quad t(u)^{\sigma} = \exp(2\widetilde{\tau}_u^{\sigma}) = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}, \quad (u \in \text{Sym}(r)),$$

$$J_r = \begin{pmatrix} 0 & -I_r \\ I_r & 0 \end{pmatrix}.$$

It is well known that the elements  $g(l_1)$ ,  $t(u)$  and  $J_r$ , for  $l_1 \in \text{GL}(r, \mathbb{R})$ ,  $u \in \text{Sym}(r, \mathbb{R})$ , generate the group symplectic group  $\text{Sp}(r, \mathbb{R})$ . One considers the two representations  $R$  and  $R^{\sigma}$  of the metaplectic group  $\text{Mp}(r, \mathbb{R})$  on  $L^2(\mathbb{R}^r)$ , determined by:

$$\begin{aligned}
R(g(l_1))f(x) &= (\det l_1)^{\frac{1}{2}} f(l_1^t x), \\
R(t(u)^\sigma)f(x) &= e^{-\frac{i}{2}\tau_u(x^2)} f(x), \\
R(J_r)f(x) &= a_0 \int_{\mathbb{R}^r} e^{i\tau(xy)} f(y) dy.
\end{aligned}$$

and

$$\begin{aligned}
R^\sigma(g(l_1))f(x) &= (\det l_1)^{-\frac{1}{2}} f(l_1^t x), \\
R^\sigma(t(u))f(x) &= e^{-\frac{i}{2}\tau_u(x^2)} f(x), \\
R^\sigma(J_r)f(x) &= a_0 \int_{\mathbb{R}^r} e^{i\tau(xy)} f(y) dy.
\end{aligned}$$

Then

$$dR(\widetilde{\tau_u^\sigma})f(x) = -\frac{i}{4}\tau_u(x^2)f(x), \quad dR(\widetilde{\tau_u})f(x) = -\frac{i}{4}\tau_u\left(\frac{\partial^2}{\partial x^2}\right)f(x)$$

and

$$dR^\sigma(\widetilde{\tau_u})f(x) = -\frac{i}{4}\tau_u(x^2)f(x), \quad dR^\sigma(\widetilde{\tau_u^\sigma})f(x) = -\frac{i}{4}\tau_u\left(\frac{\partial^2}{\partial x^2}\right)f(x)$$

In particular

$$dR(\widetilde{E})f(x) = -\frac{i}{4}\tau\left(\frac{\partial^2}{\partial x^2}\right)f(x), \quad dR(\widetilde{F})f(x) = -\frac{i}{4}\tau(x^2)f(x)$$

and

$$dR^\sigma(\widetilde{F})f(x) = -\frac{i}{4}\tau\left(\frac{\partial^2}{\partial x^2}\right)f(x), \quad dR^\sigma(\widetilde{E})f(x) = -\frac{i}{4}\tau(x^2)f(x)$$

Denote by

$$\mathrm{L}^2(\mathbb{R}^r)_{\text{even}} = \{f \in \mathrm{L}^2(\mathbb{R}^r) \mid f(-x) = f(x)\}$$

and

$$\mathrm{L}^2(\mathbb{R}^r)_{\text{odd}} = \{f \in \mathrm{L}^2(\mathbb{R}^r) \mid f(-x) = -f(x)\}.$$

The following facts are well-known:

- 1) (Irreducibility) The representations  $(R, \mathrm{L}^2(\mathbb{R}^r)_{\text{even}})$ ,  $(R, \mathrm{L}^2(\mathbb{R}^r)_{\text{odd}})$ ,  $(R^\sigma, \mathrm{L}^2(\mathbb{R}^r)_{\text{even}})$ ,  $(R^\sigma, \mathrm{L}^2(\mathbb{R}^r)_{\text{odd}})$  of  $\mathrm{Mp}(r, \mathbb{R})$  are irreducible.
- 2) ( $K$ -type decomposition) The underlying  $(\mathfrak{g}, K)$ -modules,  $(R)_{K^L}$  and  $(R_{\alpha_0}^\sigma)_{K^L}$ , for  $K = K^L = O(r, \mathbb{C})$ , have the following  $K$ -type formulas

$$\begin{aligned}
(R)_{K^L} &= \bigoplus_{m=0}^{\infty} \mathcal{Y}_{2m}(\mathbb{R}^r) + \bigoplus_{m=0}^{\infty} \mathcal{Y}_{2m+1}(\mathbb{R}^r), \\
(R^\sigma)_{K^L} &= \bigoplus_{m=0}^{\infty} \mathcal{Y}_{2m}(\mathbb{R}^r) + \bigoplus_{m=0}^{\infty} \mathcal{Y}_{2m+1}(\mathbb{R}^r).
\end{aligned}$$

- 3) (Unitarity) The representations  $R$  and  $R^\sigma$  of  $\mathrm{Mp}(r, \mathbb{R})$  on  $\mathrm{L}^2(\mathbb{R}^r)$  are unitary.

## 5. The intertwining operator

Recall from section 3 that

$$\begin{aligned} \text{Ad}(g_0^{-1})(\widetilde{\mathfrak{g}}_{\mathbb{R}}) = \\ \left\{ \begin{pmatrix} w_1 + v & w'_1 - u \\ -w'_1 - u & w_1 - v \end{pmatrix}, w_1 \in \text{Skew}(r, \mathbb{R}), w'_1, u, v \in \text{Sym}(r, \mathbb{R}) \right\} = \mathfrak{sp}(r, \mathbb{R}) \end{aligned}$$

where

$$g_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} iI_r & I_r \\ -I_r & -iI_r \end{pmatrix}.$$

Denote by  $\widetilde{G} = \text{Sp}(r, \mathbb{C})$ . It is generated by the elements  $g(l_1), t(u)$  and  $J_r$  (for  $l_1 \in \text{GL}(r, \mathbb{C}), u \in \text{Sym}(r, \mathbb{C})$ ).

The representations  $\rho := \rho_\alpha$  and  $\rho^\sigma := \rho_\alpha^\sigma$  in section 1 (i.e with  $\alpha = -\frac{1}{4}$ ) 'integrate' to the representations  $T$  and  $T^\sigma$  of  $\widetilde{G} = \text{Mp}(r, \mathbb{C})$  on  $\mathcal{O}_{\text{fin}}(\Xi)$  given by

$$T(g(l_1))\phi(z) = (\det l_1)^{2\alpha} \phi(l_1^{-1}z),$$

$$T(t(u))\phi(z) = e^{\frac{i}{2}\tau_u(z^2)}\phi(z),$$

$$T(J)\phi(z) = a_0 \int_{\mathbb{R}^r} e^{i\tau(z)y} \phi(y) dy$$

and

$$T^\sigma(g(l_1))\phi(z) = (\det l_1)^{-2\alpha} \phi(l_1^t z),$$

$$T^\sigma(t(u)^\sigma)\phi(z) = e^{\frac{i}{2}\tau_u(z^2)}\phi(z),$$

$$T^\sigma(J)\phi(z) = a_0 \int_{\mathbb{R}^r} e^{i\tau(z)y} \phi(y) dy$$

in such a way that

$$dT(\widetilde{\tau}_u^\sigma)\phi(z) = \frac{i}{4}\tau_u(z^2)\phi(z), \quad dT(\widetilde{\tau}_u)\phi(z) = \frac{i}{4}\tau_u\left(\frac{\partial^2}{\partial z^2}\right)\phi(z),$$

$$dT^\sigma(\widetilde{\tau}_u)\phi(z) = \frac{i}{4}\tau_u(z^2)\phi(z), \quad dT^\sigma(\widetilde{\tau}_u^\sigma)\phi(z) = \frac{i}{4}\tau_u\left(\frac{\partial^2}{\partial z^2}\right)\phi(z)$$

(where we precise that the exponent  $2\alpha$  arises from  $\chi(l)^\alpha = (\det l_1)^{2\alpha}$ ).

Denote by

$$\widetilde{G}_{\mathbb{R}} = \text{Ad}(g_0)(\text{Mp}(r, \mathbb{R})).$$

and consider the Hilbert spaces  $T_0^{-1}(\mathcal{F}(\mathbb{C}^r))$  and  $T_0^{\sigma^{-1}}(\mathcal{F}(\mathbb{C}^r))$ , equipped with the norms

$$\|\psi\|_{T_0^{-1}(\mathcal{F}(\mathbb{C}^r))} = \|T_0\psi\|_{\mathcal{F}(\mathbb{C}^r)}$$

and

$$\|\psi\|_{T_0^{\sigma^{-1}}(\mathcal{F}(\mathbb{C}^r))} = \|T_0^{\sigma}\psi\|_{\mathcal{F}(\mathbb{C}^r)}.$$

Then,  $(T, T(g_0^{-1})(\mathcal{F}(\mathbb{C}^r)))$  and  $(T^{\sigma}, T^{\sigma}(g_0^{-1})(\mathcal{F}(\mathbb{C}^r)))$  are unitary representations of the metaplectic group  $\text{Mp}(r, \mathbb{R})$ . In fact, for  $g \in \text{Mp}(r, \mathbb{R})$ , there is  $g' \in \widetilde{G}_{\mathbb{R}}$  such that

$$g = g_0^{-1}g'g_0,$$

then

$$T(g) = T(g_0^{-1}T(g')T(g_0)) \text{ and } T^{\sigma}(g) = T^{\sigma}(g_0^{-1}T^{\sigma}(g')T^{\sigma}(g_0)).$$

It follows that for  $\psi \in T_0^{-1}(\mathcal{F}(\mathbb{C}^r))$ ,  $\psi^{\sigma} \in (T_0^{\sigma})^{-1}(\mathcal{F}(\mathbb{C}^r))$  and for  $g \in \text{Mp}(r, \mathbb{R})$ ,

$$\begin{aligned} \|T(g)\psi\|_{T(g_0^{-1})(\mathcal{F}(\mathbb{C}^r))} &= \|T(g_0^{-1})T(g')T(g_0)\psi\|_{T(g_0^{-1})(\mathcal{F}(\mathbb{C}^r))} \\ &= \|T(g')T(g_0)\psi\|_{\mathcal{F}(\mathbb{C}^r)} = \|T(g_0)\psi\|_{\mathcal{F}(\mathbb{C}^r)} = \|\psi\|_{T(g_0^{-1})(\mathcal{F}(\mathbb{C}^r))} \end{aligned}$$

and similarly,

$$\begin{aligned} \|T^{\sigma}(g)\psi^{\sigma}\|_{T^{\sigma}(g_0^{-1})(\mathcal{F}(\mathbb{C}^r))} &= \|T^{\sigma}(g_0^{-1})T^{\sigma}(g')T^{\sigma}(g_0)\psi^{\sigma}\|_{T^{\sigma}(g_0^{-1})(\mathcal{F}(\mathbb{C}^r))} \\ &= \|T^{\sigma}(g')T^{\sigma}(g_0)\psi^{\sigma}\|_{\mathcal{F}(\mathbb{C}^r)} = \|T^{\sigma}(g_0)\psi^{\sigma}\|_{\mathcal{F}(\mathbb{C}^r)} = \|\psi^{\sigma}\|_{T^{\sigma}(g_0^{-1})(\mathcal{F}(\mathbb{C}^r))}. \end{aligned}$$

Recall that the Bargmann transform

$$\mathcal{B} : L^2(\mathbb{R}^r) \rightarrow \mathcal{F}(\mathbb{C}^r)$$

is a unitary operator given by the integral formula

$$(\mathcal{B}f)(z) = \pi^{-\frac{r}{4}} \int_{\mathbb{R}^r} e^{-\frac{1}{2}(\tau(x^2) + \tau(z^2)) + \sqrt{2}\tau(zx)} f(x) dx$$

where  $dx$  is Lebesgue measure, and  $\tau$  is the usual symmetric bilinear form in  $r$  variables.

**Theorem 5.1—**

(i) The unitary representations  $(R, L^2(\mathbb{R}^r))$  and  $(T^\sigma, T^\sigma(g_0^{-1})(\mathcal{F}(\mathbb{C}^r)))$  of the group  $\text{Mp}(r, \mathbb{R})$  are unitarily equivalent. The intertwining operator is given by  $\mathcal{B}_0 = T(g_0^{-1}) \circ \mathcal{B}$ .

(ii) The unitary representations  $(R^\sigma, L^2(\mathbb{R}^r))$  and  $(T, T(g_0^{-1})(\mathcal{F}(\mathbb{C}^r)))$  of the group  $\text{Mp}(r, \mathbb{R})$  are unitarily equivalent. The intertwining operator is given by  $\mathcal{B}_0^\sigma = T^\sigma(g_0^{-1}) \circ \mathcal{B}$ .

**Proof.**

In fact, if an operator  $\mathcal{B}_0 : L^2(\mathbb{R}^r) \rightarrow T^\sigma(g_0^{-1})(\mathcal{F}(\mathbb{C}^r))$  intertwins  $R$  and  $T^\sigma$ , then for every  $g = g_0^{-1}g'g_0 \in \text{Mp}(r, \mathbb{R})$ , one has

$$T^\sigma(g)\mathcal{B}_0 = \mathcal{B}_0 R(g),$$

i.e.

$$T^\sigma(g_0^{-1})T^\sigma(g')T^\sigma(g_0)\mathcal{B}_0 = \mathcal{B}_0 R(g)$$

which means

$$T^\sigma(g')(T^\sigma(g_0)\mathcal{B}_0) = (T^\sigma(g_0)\mathcal{B}_0)R(g)$$

i.e.

$$T^\sigma(g')\tilde{\mathcal{B}} = \tilde{\mathcal{B}}R(g) \text{ with } \tilde{\mathcal{B}} = (T^\sigma(g_0)\mathcal{B}_0).$$

Similarly, if an operator  $\mathcal{B}_0^\sigma : L^2(\mathbb{R}^r) \rightarrow T(g_0^{-1})(\mathcal{F}(\mathbb{C}^r))$  intertwins  $R^\sigma$  and  $T$ , then for every  $g = g_0^{-1}g'g_0 \in \text{Mp}(r, \mathbb{R})$ , one has

$$T(g)\mathcal{B}_0^\sigma = \mathcal{B}_0^\sigma R^\sigma(g),$$

i.e.

$$T(g_0^{-1})T(g')T(g_0)\mathcal{B}_0^\sigma = \mathcal{B}_0^\sigma R^\sigma(g)$$

which means

$$T(g')(T(g_0)\mathcal{B}_0^\sigma) = (T(g_0)\mathcal{B}_0^\sigma)R^\sigma(g)$$

i.e.

$$T(g')\tilde{\mathcal{B}}^\sigma = \tilde{\mathcal{B}}^\sigma R^\sigma(g) \text{ with } \tilde{\mathcal{B}}^\sigma = (T(g_0)\mathcal{B}_0^\sigma).$$

In what follows, we will see that  $\tilde{\mathcal{B}} = \tilde{\mathcal{B}}^\sigma = \mathcal{B}$ , the classical Bargmann transform. In fact, let  $b$  and  $b^\sigma$  be functions in one complex variable and let  $\tilde{\mathcal{B}} : L^2(\mathbb{R}^r) \rightarrow \mathcal{F}(\mathbb{C}^r)$  and  $\tilde{\mathcal{B}}^\sigma : L^2(\mathbb{R}^r) \rightarrow \mathcal{F}(\mathbb{C}^r)$  be the integral operators given by : for  $x \in S$ ,  $\xi_z = zz^t \in \Gamma(\Xi)$ ,

$$(\tilde{\mathcal{B}}f)(\xi_z) = \int_S b(z, x)f(x)s(dx)$$

and

$$(\tilde{\mathcal{B}}^\sigma f)(\xi) = \int_S b^\sigma(z, x)f(x)s(dx).$$

They map  $\mathcal{C}^\infty(S)$  into  $\mathcal{O}(\Gamma(\Xi))$ , are  $O(r)$ -equivariant, and map  $\mathcal{Y}_k(\mathbb{R}^r)$  onto  $\tilde{\mathcal{O}}_k(\mathbb{C}^r)$ .

The 'intertwining' relation for  $\tilde{\mathcal{B}}^\sigma$ :

$$T(g')\tilde{\mathcal{B}}^\sigma = \tilde{\mathcal{B}}^\sigma R^\sigma(g)$$

leads in particular to the 'intertwining' relations

$$dT(\tilde{E} + \tilde{F}) \circ \tilde{\mathcal{B}}^\sigma = \tilde{\mathcal{B}}^\sigma \circ dR^\sigma(\text{Ad}(g_0^{-1})(\tilde{E} + \tilde{F})),$$

and

$$dT(i(\tilde{E} - \tilde{F})) \circ \tilde{\mathcal{B}}^\sigma = \tilde{\mathcal{B}}^\sigma \circ dR^\sigma(\text{Ad}(g_0^{-1})(i(\tilde{E} - \tilde{F}))).$$

But, since

$$\tilde{E} = \frac{1}{2} \begin{pmatrix} 0 & 0 \\ I_r & 0 \end{pmatrix}, \quad \tilde{F} = \frac{1}{2} \begin{pmatrix} 0 & I_r \\ 0 & 0 \end{pmatrix},$$

then

$$\tilde{E} + \tilde{F} = \frac{1}{2} \begin{pmatrix} 0 & I_r \\ I_r & 0 \end{pmatrix} \text{ and } i(\tilde{E} - \tilde{F}) = \frac{1}{2} \begin{pmatrix} 0 & -iI_r \\ iI_r & 0 \end{pmatrix},$$

and

$$\text{Ad}(g_0^{-1})(\tilde{E} + \tilde{F}) = \frac{1}{2} \begin{pmatrix} 0 & -I_r \\ -I_r & 0 \end{pmatrix} = -(\tilde{E} + \tilde{F}),$$

$$\text{Ad}(g_0^{-1})(i(\tilde{E} - \tilde{F})) = \frac{1}{2} \begin{pmatrix} I_r & 0 \\ 0 & -I_r \end{pmatrix} =: \frac{1}{2}\omega(I_r).$$

The 'intertwining' relations become

$$dT(\tilde{E}) \circ \tilde{\mathcal{B}}^\sigma + dT(\tilde{F}) \circ \tilde{\mathcal{B}}^\sigma = -\tilde{\mathcal{B}}^\sigma \circ dR^\sigma(\tilde{E}) - \tilde{\mathcal{B}}^\sigma \circ dR^\sigma(\tilde{F}),$$

and

$$idT(\tilde{E}) \circ \tilde{\mathcal{B}}^\sigma - idT(\tilde{F}) \circ \tilde{\mathcal{B}}^\sigma = \tilde{\mathcal{B}}^\sigma \circ dR^\sigma(\frac{1}{2}\omega(I_r)).$$

It follows that

$$2dT(\tilde{F}) \circ \tilde{\mathcal{B}}^\sigma = \tilde{\mathcal{B}}^\sigma \circ (\frac{i}{2}dR^\sigma(\omega(I_r)) + dR^\sigma(-\tilde{E}) + dR^\sigma(-\tilde{F})) \quad (*).$$

Finally, using the integral form for the operator  $\tilde{\mathcal{B}}^\sigma$ , and the formulas

$$dR^\sigma(\omega(I_r))f(x) = dR^\sigma\left(\begin{pmatrix} I_r & 0 \\ 0 & -I_r \end{pmatrix}\right)f(x) = -\frac{r}{2}f(x) + \mathcal{E}f(x) \text{ for } f \in L^2(\mathbb{R}^r),$$

$$dR^\sigma(\tilde{E})f(x) = -\frac{i}{4}\tau\left(\frac{\partial^2}{\partial x^2}\right)f(x), \quad dR^\sigma(\tilde{F})f(x) = -\frac{i}{4}\tau(x^2)f(x) \text{ for } f \in L^2(\mathbb{R}^r),$$

and

$$dT(\tilde{E})\phi(z) = \frac{i}{4}\tau(z^2)\phi(z), \quad dT(\tilde{F})\phi(z) = \frac{i}{4}\tau\left(\frac{\partial^2}{\partial z^2}\right)\phi(z) \text{ for } \phi \in \mathcal{F}(\mathbb{C}^r),$$



one can deduce from (\*) that

$$\begin{aligned}
& \frac{i}{2}\tau\left(\frac{\partial^2}{\partial z^2}\right) \int_{\mathbb{R}^r} b^\sigma(z, x) f(x) dx \\
&= \int_{\mathbb{R}^r} b^\sigma(z, x) \left( -\frac{ir}{4} + \frac{i}{2}\tau\left(x\frac{\partial}{\partial x}\right) + \frac{i}{4}\tau\left(\frac{\partial^2}{\partial x^2}\right) + \frac{i}{4}\tau(x^2) \right) f(x) dx \\
&= \int_{\mathbb{R}^r} \left( -\frac{ir}{4} - \frac{i}{2}\tau\left(x\frac{\partial}{\partial x}\right) + \frac{i}{4}\tau\left(\frac{\partial^2}{\partial x^2}\right) + \frac{i}{4}\tau(x^2) \right) b^\sigma(z, x) f(x) dx
\end{aligned}$$

which leads to the following differential equation for the function  $b^\sigma$ :

$$-\tau\left(\frac{\partial^2}{\partial z^2}\right)b^\sigma(z, x) = \left(\frac{r}{2} + \tau\left(x\frac{\partial}{\partial x}\right) - \frac{1}{2}\tau\left(\frac{\partial^2}{\partial x^2}\right) - \frac{1}{2}\tau(x^2)\right)b^\sigma(z, x).$$

Observe that the solution of this equation is given by:

$$b^\sigma(z, x) = e^{-\frac{1}{2}(\tau(x^2) + \tau(z^2)) + \sqrt{2}\tau(zx)}.$$

In fact, one has

$$\begin{aligned}
\frac{\partial^2}{\partial z_i^2} b(z, x) &= \frac{\partial}{\partial z_i} (-z_i + \sqrt{2}x_i) b^\sigma(z, x) \\
&= (-1 + (-z_i + \sqrt{2}x_i)^2) b^\sigma(z, x) \\
&= (-1 + z_i^2 + 2x_i^2 - 2\sqrt{2}x_i z_i) b^\sigma(z, x).
\end{aligned}$$

Then

$$-\tau\left(\frac{\partial^2}{\partial z^2}\right)b^\sigma(z, x) = (r - \tau(z^2) - 2\tau(x^2) + 2\sqrt{2}\tau(xz))b^\sigma(z, x).$$

Similarly, one gets

$$-\tau\left(\frac{\partial^2}{\partial x^2}\right)b^\sigma(z, x) = (r - \tau(x^2) - 2\tau(z^2) + 2\sqrt{2}\tau(xz))b^\sigma(z, x).$$

On another part,

$$x_i \frac{\partial}{\partial x_i} b^\sigma(z, x) = -x_i^2 + \sqrt{2}x_i z_i.$$

Then

$$\tau\left(x\frac{\partial}{\partial x}\right)b^\sigma(z, x) = (-\tau(x^2) + \sqrt{2}\tau(xz))b^\sigma(z, x).$$

It follows that

$$\begin{aligned}
& \left(\frac{r}{2} + \tau\left(x\frac{\partial}{\partial x}\right) - \frac{1}{2}\tau\left(\frac{\partial^2}{\partial x^2}\right) - \frac{1}{2}\tau(x^2)\right)b^\sigma(z, x) \\
&= \left(\frac{r}{2} - \tau(x^2) + \sqrt{2}\tau(zx) + \frac{r}{2} - \frac{1}{2}\tau(x^2) - \tau(z^2) + \sqrt{2}\tau(zx) - \frac{1}{2}\tau(x^2)\right)b^\sigma(z, x) \\
&= (r - 2\tau(x^2) - \tau(z^2) + 2\sqrt{2}\tau(zx))b^\sigma(z, x). \quad \square
\end{aligned}$$

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